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The classification of the Ricci tensor in general relativity theory

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Abstract. Tangent space null rotations are used to give a straightforward classification of the Ricci tensor in general relativity theory.

Several discussions of the algebraic classification of the Ricci tensor in general relativity have been given (for example Churchill 1932, Plebanski 1964, Plebanski and Stachel 1968, Barnes 1974). In this note, a straightforward approach to such a classification is presented which makes use of the null rotation subgroup of the proper orthochronous Lorentz group. Although the Ricci tensor is singled out, the results given apply to any symmetric second-order tensor on a four-dimensional Lorentzian manifold.

Throughout, M will denote a space–time (a four-dimensional Lorentzian manifold) and if $p \in M$, $T_p(M)$ will denote the tangent space to M at p—a four-dimensional real Lorentzian inner product space. Suppose that a basis for $T_p(M)$ is chosen so that the components of a certain null vector $\boldsymbol{l} \in T_p(M)$ are l^i. Then $\boldsymbol{l}$ can be supplemented by means of another null vector and two spacelike vectors in $T_p(M)$ whose components in this basis are m^i, a^i, b^i respectively† and such that these four vectors form a null tetrad satisfying $l^i m_i = a^i a_i = b^i b_i = 1$, with all other inner products zero. If the components of the metric tensor in this basis are g_{ij}, one has the completeness relation $g_{ij} = 2l_{(i}m_{j)} + a_i a_j + b_i b_j$. Of the proper orthochronous Lorentz group of linear transformations of $T_p(M)$ into itself, the subgroup of null rotations about $\boldsymbol{l}$ will be particularly useful. The members of this subgroup preserve the direction of $\boldsymbol{l}$ and are conveniently represented in terms of components in the above basis by (Sachs 1961)

$$
\begin{aligned}
l^i &\to l'^i = A l^i \\
a^i &\to a'^i = \alpha a^i - \beta b^i - A\sqrt{2}(\alpha\gamma + \beta\delta) l^i \\
b^i &\to b'^i = \beta a^i + \alpha b^i + A\sqrt{2}(\alpha\delta - \beta\gamma) l^i \\
m^i &\to m'^i = A^{-1} m^i + \sqrt{2}(\gamma a^i - \delta b^i) - A(\gamma^2 + \delta^2) l^i
\end{aligned}
\qquad (1)
$$

where $A, \alpha, \beta, \gamma, \delta$ are real parameters, arbitrary except for the restrictions $\alpha^2 + \beta^2 = 1$ and $A > 0$.

† Latin indices take the values 0, 1, 2, 3, the summation convention is used throughout and round brackets denote the usual symmetrization.

The covariant Ricci tensor with components R_{ij} can be represented in terms of the members of the null tetrad, with the aid of the completeness relation, by the formula

$$R_{ij} = 2R^1 l_{(i}m_{j)} + R^2 l_i l_j + R^3 m_i m_j + 2R^4 l_{(i}a_{j)} + 2R^5 l_{(i}b_{j)} + 2R^6 m_{(i}a_{j)} + 2R^7 m_{(i}b_{j)} + 2R^8 a_{(i}b_{j)} + R^9 a_i a_j + R^{10} b_i b_j. \qquad (2)$$

The mixed Ricci tensor with components $R^i_{\ j}$ can be represented as a linear transformation $R: T_p(M) \to T_p(M)$ with matrix $R^i_{\ j}$. The following general results about R are available (cf Churchill 1932).

(i) There always exists a two-dimensional subspace of $T_p(M)$ which is an invariant 2-space of R.

(ii) If V is an invariant 2-space of R then so is the 2-space orthogonal to V.

(iii) R has a spacelike invariant 2-space (equivalently a timelike invariant 2-space)$\Leftrightarrow R$ has two distinct spacelike eigenvectors.

(iv) R has a null invariant 2-space$\Leftrightarrow R$ has a null eigenvector.

To prove these results it is pointed out that a two-dimensional subspace of $T_p(M)$ is called *timelike*, *null* or *spacelike* according as it contains exactly two, one or no null vectors. The families of timelike, null and spacelike 2-spaces partition the family of all 2-spaces of $T_p(M)$. Also the 2-space orthogonal to a spacelike (respectively timelike, null) 2-space of $T_p(M)$ is timelike (respectively spacelike, null). To prove (i), note that if R^j_i is similar to a Jordan matrix, then the first two members (in the conventional numbering) of a Jordan basis for R^j_i span an invariant 2-space of R. If R^j_i is not similar to a Jordan matrix, the real and imaginary parts of a complex eigenvector of R span an invariant 2-space of R. The proof of (ii), (iii) and (iv) follows from the following remarks. If R has a spacelike invariant 2-space spanned by orthogonal unit spacelike vectors a^i and b^i, construct a null tetrad with components l^i, m^i, a^i, b^i. In terms of this null tetrad, R_{ij} takes the form (2) with $R^4 = R^5 = R^6 = R^7 = 0$ whence l^i and m^i span an orthogonal timelike invariant 2-space. The proof when R has a timelike invariant 2-space is similar. In the case when R has a spacelike invariant 2-space spanned by unit spacelike vectors a^i and b^i, one can start from the above null tetrad and perform a null rotation (1) with $A = 1$, $\gamma = \delta = 0$ and α, β chosen such that in the new null tetrad $\acute{l}^i\, \acute{m}^i\, \acute{a}^i\, \acute{b}^i$, $\acute{R}^4 = \acute{R}^5 = \acute{R}^6 = \acute{R}^7 = \acute{R}^8 = 0$. It then follows that $\acute{a}^i$ and $\acute{b}^i$ are the components of distinct spacelike eigenvectors of R. If R has a null invariant 2-space, let l^i, a^i span this 2-space, where $l^i l_i = l^i a_i = 0$ and a^i is a unit spacelike vector. One now constructs a null tetrad $l^i\, m^i\, a^i\, b^i$ and in this tetrad R_{ij} takes the form (2) with $R^3 = R^6 = R^7 = R^8 = 0$. Thus l^i is an eigenvector of R and l^i and b^i span a null invariant 2-space orthogonal to the original one. All the results above now follow easily. The results (i) and (ii) show that at least two invariant 2-spaces of R always exist.

The various canonical forms for the Ricci tensor, based on equation (2), can now be derived. Two cases are considered.

Case A. The Ricci tensor has a null eigenvector

Let l^i be the components of a null eigenvector of the Ricci tensor. If a null tetrad containing l^i is constructed with members l^i, m^i, a^i, b^i, the Ricci tensor can be written in the form (2) with $R^3 = R^6 = R^7 = 0$. One now simplifies the expression (2) by performing a null rotation about l^i of the form (1) with $A = 1$, $\gamma = \delta = 0$ and α and β chosen so that in the new null tetrad $\acute{l}^i\, \acute{m}^i\, \acute{a}^i\, \acute{b}^i$, $\acute{R}^3 = \acute{R}^6 = \acute{R}^7 = \acute{R}^8 = 0$. Four possibilities now arise: (*a*) $\acute{R}^1 \neq \acute{R}^9, \acute{R}^1 \neq \acute{R}^{10}$, (*b*) $\acute{R}^1 = \acute{R}^9 \neq \acute{R}^{10}$, (*c*) $\acute{R}^1 = \acute{R}^{10} \neq \acute{R}^9$ and (*d*) $\acute{R}^1 = \acute{R}^9 =$

$\acute{R}^{10}$. In case (a) one can perform a null rotation (1) with $\alpha = A = 1$, $\beta = 0$ and with γ and δ chosen so that in the new null tetrad, $\acute{\acute{R}}^3 = \acute{\acute{R}}^4 = \acute{\acute{R}}^5 = \acute{\acute{R}}^6 = \acute{\acute{R}}^7 = \acute{\acute{R}}^8 = 0$. So the Ricci tensor takes the form

$$R_{ij} = 2\rho_1 l_{(i}m_{j)} + \lambda l_i l_j + \rho_2 a_i a_j + \rho_3 b_i b_j \tag{3}$$

where the primes have been omitted from the tetrad vectors. If $\lambda \neq 0$, the vectors l^i, a^i and b^i are eigenvectors of the Ricci tensor with eigenvalues ρ_1, ρ_2 and ρ_3 respectively. In this case it is easily shown that R_i^j is similar to a Jordan matrix with Segré type {(2)(1)(1)} or {(2)(1, 1)}. If $\lambda = 0$, l^i, m^i, a^i and b^i are all eigenvectors of the Ricci tensor with eigenvalues ρ_1, ρ_1, ρ_2 and ρ_3 respectively. In this case R_i^j is similar to a diagonal matrix and has Segré type {(1, 1)(1)(1)} or {(1, 1)(1, 1)}. The cases (b) and (c) are similar. In case (b), starting from the initial null tetrad in which $\acute{R}^3 = \acute{R}^6 = \acute{R}^7 = \acute{R}^8 = 0$, if also $\acute{R}^4 = 0$ then there exists a null rotation (1) with $A = \alpha = 1$, $\beta = \gamma = 0$, yielding a new tetrad with $\acute{\acute{R}}^3 = \acute{\acute{R}}^4 = \acute{\acute{R}}^5 = \acute{\acute{R}}^6 = \acute{\acute{R}}^7 = \acute{\acute{R}}^8 = 0$ and equation (3) together with the Segré types {(2, 1)(1)} or {(1, 1, 1)(1)} applies. If in the initial null tetrad, $\acute{R}^4 \neq 0$, then there exists a null rotation with $A = \alpha = 1$, $\beta = 0$ yielding a null tetrad in which $\acute{\acute{R}}^2 = \acute{\acute{R}}^3 = \acute{\acute{R}}^5 = \acute{\acute{R}}^6 = \acute{\acute{R}}^7 = \acute{\acute{R}}^8 = 0$ and (on omitting primes again) a canonical form

$$R_{ij} = 2\rho_1 l_{(i}m_{j)} + 2\sigma l_{(i}a_{j)} + \rho_1 a_i a_j + \rho_2 b_i b_j. \tag{4}$$

Since $\acute{R}^4 \neq 0$, we have $\sigma \neq 0$ and so l^i and b^i are eigenvectors of the Ricci tensor with eigenvalues ρ_1 and ρ_2 respectively. The matrix R_i^j is similar to a Jordan matrix with Segré type {(3)(1)}. Case (c) is similar. In case (d), if $\acute{R}^4 = \acute{R}^5 = 0$, equation (3) again results with Segré types {(2, 1, 1)} or {(1, 1, 1, 1}. If $(\acute{R}^4)^2 + (\acute{R}^5)^2 \neq 0$, a null rotation with $A = 1$ can be used to obtain a null tetrad in which the canonical form (4) holds with $\rho_1 = \rho_2$. In this case one again has $\sigma \neq 0$ and R_i^j is similar to a Jordan matrix with Segré type {(3, 1)}.

Case B. The Ricci tensor has no null eigenvectors

If the Ricci tensor has no null eigenvectors, it follows from results (i), (ii), (iii) and (iv) above that it has two distinct spacelike eigenvectors with, say, components a^i and b^i. On constructing a null tetrad with components l^i, m^i, a^i, b^i, the Ricci tensor in this tetrad takes the form (2) with $R^4 = R^5 = R^6 = R^7 = R^8 = 0$. The condition that the Ricci tensor has no null eigenvectors implies that $R^2 \neq 0$, $R^3 \neq 0$. The ambiguity in this null tetrad, represented by a null rotation (1) with $\alpha = 1$, $\gamma = \delta = \beta = 0$ and A arbitrary, can be utilized to ensure that $|R^2| = |R^3|$. The two possibilities $R^2 = R^3$ and $R^2 = -R^3$ yield the respective canonical forms:

$$R_{ij} = 2\rho_1 l_{(i}m_{j)} + \rho_2(l_i l_j + m_i m_j) + \rho_3 a_i a_j + \rho_4 b_i b_j \tag{5}$$

$$R_{ij} = 2\rho_5 l_{(i}m_{j)} + \rho_6(l_i l_j - m_i m_j) + \rho_7 a_i a_j + \rho_8 b_i b_j \tag{6}$$

where the condition $\rho_3 \neq \rho_1 - \rho_2 \neq \rho_4$ holds in equation (5). Equation (5) shows that R_i^j is similar to a diagonal matrix with Segré type {(1)(1)(1)(1)} or some degeneracy of this type. There is one timelike eigenvector $l^i - m^i$ and three spacelike eigenvectors $l^i + m^i$, a^i and b^i and their eigenvalues are respectively $\rho_1 - \rho_2$, $\rho_1 + \rho_2$, ρ_3 and ρ_4. If the Ricci tensor takes the form (6), then the real eigenvectors are a^i and b^i with (real) eigenvalues ρ_7 and ρ_8 respectively. In this case, R_i^j is not similar to a Jordan matrix but rather has complex eigenvectors $l^i \pm i m^i$ with corresponding eigenvalues $\rho_5 \pm i\rho_6$.

Some concluding remarks can now be given. Firstly, it follows immediately from equation (2) that if a Ricci tensor has two distinct null eigendirections, then their corresponding eigenvalues are equal and R_i^j is similar to a diagonal matrix. (The equality of the eigenvalues also follows from the fact that two eigenvectors with different eigenvalues are orthogonal.) The results given above also show that a Ricci tensor has a unique null eigendirection$\Leftrightarrow R_i^j$ is similar to a Jordan matrix with a non-simple elementary divisor, and that R_i^j is diagonable$\Leftrightarrow$the Ricci tensor has a timelike eigenvector. All Ricci tensors have at least two distinct real eigenvectors. Secondly, it is easily shown that the null tetrad yielding the canonical structure (4) is uniquely determined when the eigenvalues ρ_1 and ρ_2 are distinct and when the value of σ is to be preserved. However, in equation (3) with $\lambda \neq 0$, even when the eigenvalues ρ_1, ρ_2 and ρ_3 are distinct and the value of λ to be preserved, the null tetrad is not unique since alternative tetrads are obtainable from (1) with $A = 1$, $\gamma = \delta = 0$ and either $\beta = 0$, $\alpha = -1$ or $\alpha = 0$, $\beta = \pm 1$. Similar ambiguities arise in equations (5) and (6) when the eigenvalues are distinct. Further ambiguities of tetrads are introduced in the event of certain eigenvalues being equal. Also†, in (3) with $\lambda \neq 0$, for a given tetrad and triple of

Table 1.

Present notation	*Segré type*	*Plebanski type*
Case A. The Ricci tensor has a null eigenvector		
Subcase (a)		
$\acute{R}^1 \neq \acute{R}^9 \neq \acute{R}^{10} \neq \acute{R}^1$, $\lambda \neq 0$	$\{(2)(1)(1)\}$	$[2N-S_1-S_2]_{[2-1-1]}$
$\acute{R}^1 \neq \acute{R}^9 \neq \acute{R}^{10} \neq \acute{R}^1$, $\lambda = 0$	$\{(1,1)(1)(1)\}$	$[2T-S_1-S_2]_{[1-1-1]}$
$\acute{R}^1 \neq \acute{R}^9 = \acute{R}^{10} \neq \acute{R}^1$, $\lambda \neq 0$	$\{(2)(1,1)\}$	$[2N-2S]_{[2-1]}$
$\acute{R}^1 \neq \acute{R}^9 = \acute{R}^{10} \neq \acute{R}^1$, $\lambda = 0$	$\{(1,1)(1,1)\}$	$[2T-2S]_{[1-1]}$
Subcase (b) (case (c) similar)		
$\acute{R}^1 = \acute{R}^9 \neq \acute{R}^{10}$ $(\acute{R}^4 = 0 \quad \lambda \neq 0)$	$\{(2,1)(1)\}$	$[3N-S]_{[2-1]}$
$\acute{R}^1 = \acute{R}^9 \neq \acute{R}^{10}$ $(\acute{R}^4 = 0 \quad \lambda = 0)$	$\{(1,1,1)(1)\}$	$[3T-S]_{[1-1]}$
$\acute{R}^1 = \acute{R}^9 \neq \acute{R}^{10}$ $(\acute{R}^4 \neq 0)$	$\{(3)(1)\}$	$[3N-S]_{[3-1]}$
Subcase (d)		
$\acute{R}^1 = \acute{R}^9 = \acute{R}^{10}$ $(\acute{R}^4 = \acute{R}^5 = 0 \quad \lambda \neq 0)$	$\{(2,1,1)\}$	$[4N]_{[2]}$
$\acute{R}^1 = \acute{R}^9 = \acute{R}^{10}$ $(\acute{R}^4 = \acute{R}^5 = 0 \quad \lambda = 0)$	$\{(1,1,1,1)\}$	$[4T]_{[1]}$
$\acute{R}^1 = \acute{R}^9 = \acute{R}^{10}$ $((\acute{R}^4)^2 + (\acute{R}^5)^2) \neq 0)$	$\{(3,1)\}$	$[4N]_{[3]}$
Case B. The Ricci tensor has no null eigenvectors		
$R^2 = R^3$, $\rho_1 + \rho_2 = \rho_3 = \rho_4$	$\{(1)(1,1,1)\}$	$[T-3S]_{[1-1]}$
$R^2 = R^3$, $\rho_1 + \rho_2 = \rho_3 \neq \rho_4$	$\{(1)(1)(1,1)\}$	$[T-S_1-2S_2]_{[1-1-1]}$
$R^2 = R^3$, $\rho_1 + \rho_2 = \rho_4 \neq \rho_3$	$\{(1)(1)(1,1)\}$	$[T-S_1-2S_2]_{[1-1-1]}$
$R^2 = R^3$, $\rho_1 + \rho_2 \neq \rho_3 = \rho_4$	$\{(1)(1)(1,1)\}$	$[T-S_1-2S_2]_{[1-1-1]}$
$R^2 = R^3$, $\rho_3 \neq \rho_4 \neq \rho_1 + \rho_2 \neq \rho_3$	$\{(1)(1)(1)(1)\}$	$[T-S_1-S_2-S_3]_{[1-1-1-1]}$
$R^2 = -R^3$, $\rho_7 = \rho_8$	$\{Z\bar{Z}(1,1)\}$	$[Z-\bar{Z}-2S]_{[1-1-1]}$
$R^2 = -R^3$, $\rho_7 \neq \rho_8$	$\{Z\bar{Z}(1)(1)\}$	$[Z-\bar{Z}-S_1-S_2]_{[1-1-1-1]}$

† Here, the problem of determining the distinct Lorentz orbits is touched upon. A complete list of such orbits has been given by Collinson and Shaw (1972).

eigenvalues ρ_1, ρ_2 and ρ_3 corresponding to the eigenvectors l^i, a^i and b^i, two different Ricci tensors are determined (distinguished by the sign of λ) in the following sense; any such Ricci tensor may be reduced by a transformation (1) to the form (3) in which the coefficient of $l_i l_j$ is $+1(-1) \Leftrightarrow \lambda > 0$ ($\lambda < 0$) in the original tetrad. In contrast, any Ricci tensor satisfying (4) ($\sigma \neq 0$) may be reduced to the form (4) with $\sigma = 1$ by a transformation of the form (1). Thirdly, a classification of the Ricci tensor can be achieved directly by considering the possible Jordan and rational forms for a 4×4 matrix. Since the real numbers are not algebraically closed, the rational form is introduced to deal with those cases when complex eigenvalues occur. Since this approach (or variations of it) considers the matrices R_i^j and g_i^j, the signature of the metric is disregarded and the Lorentzian signature must be imposed as a constraint after the algebraic results have been established. As a result, certain canonical types are ruled out as being inconsistent with the Lorentzian signature of g_{ij} and the symmetry of R_{ij} (cf Plebanski 1964, Barnes 1974). The present approach incorporates the Lorentzian signature from the beginning and no elimination procedure is necessary. Fourthly, for any Ricci tensor which has non-real eigenvalues or has Segré type {(3)(1)} or {(3, 1)}, the canonical forms (4) and (6) can be used to show the existence of timelike vectors with components u^i, v^i, say, satisfying $R_{ij}u^iu^j < 0 < R_{ij}v^iv^j$ and so such Ricci tensors would not be considered physically significant in general relativity because of the well known 'energy condition'†. Finally, table 1 gives a summary of the present classification and the associated Segré types and relates it to the well known Plebanski classification of the Ricci tensor (Plebanski 1964).

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† The author's attention has recently been drawn to the paper by Collinson and Shaw (1972) in which this result appears.